W-2/17

EECS 16B Section 5B

Main Topic: Phasor Analysis

Administrivia:

- HW 5 due Fri, 2/19
- Anonymous Feedback:
  bit.ly/maxwell-16B-feedback-sp21

Agenda:

- Sinusoids
- Phasors
  - Motivation
  - Derivation
- Impedance
- Q1: Phasor Analysis
- Q2: RLC Circuit Phasor Analysis

16A: Constant voltage V, current I

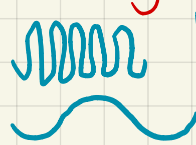
16B: Time-varying voltage $v(t)$, current $i(t)$

We consider sinusoidal voltages and currents of a specific form:

Voltage	$v(t) = V_0 \cos(\omega t + \phi_v)$
Current	$i(t) = I_0 \cos(\omega t + \phi_i)$

"phi"

"omega"



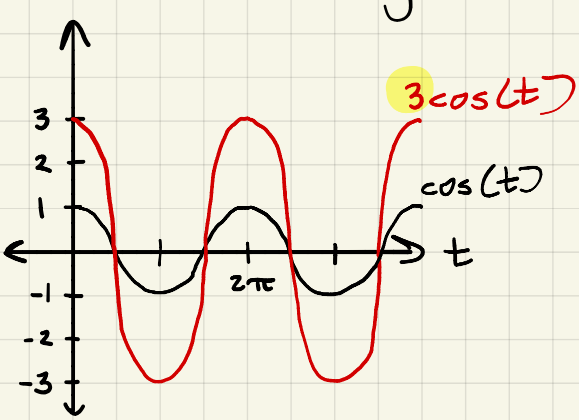
high ω

low ω

$$v(t) = V_0 \cos(\omega t + \phi)$$

V_0 = Voltage Amplitude

- Scales maximum value of function / signal



ω = angular frequency

- $\omega = 2\pi f$
- $f = \frac{1}{T}$ where T = period of sinusoid

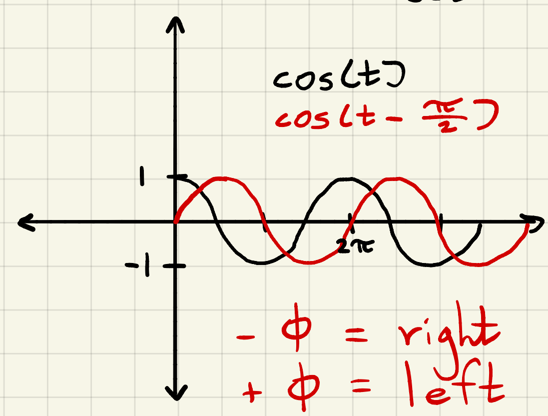
$T = 2\pi$

$f = \frac{1}{2\pi}$

$\omega = 2\pi f = 1$

ϕ = phase

- time delay aka offset



$-\phi$ = right

$+\phi$ = left

Motivation for Phasors:

- Differential Equations are <u>annoying</u>
- Quantities change over time

$e^{j\theta} = \cos\theta + j\sin\theta \Rightarrow Re\{e^{j\theta}\} = \cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$

$\cos\theta \leftrightarrow Re\{e^{j\theta}\}$, $\sin\theta \leftrightarrow Im\{e^{j\theta}\}$

Euler's Formula - See Note j

Now, consider $V_0 \cos(\omega t + \phi)$ $[\omega t + \phi = \text{"}\theta\text{"}]$

$$= \frac{V_0}{2} e^{j(\omega t + \phi)} + \frac{V_0}{2} e^{-j(\omega t + \phi)}$$

$$= \boxed{\frac{V_0}{2} e^{j\phi}} \cancel{e^{j\omega t}} + \cancel{\frac{V_0}{2} e^{-j\phi} e^{-j\omega t}}$$

conjugates

$e^{\alpha + \beta} = e^{\alpha} e^{\beta}$

Alt:

$V_0 Re\{\frac{1}{2} e^{j(\omega t + \phi)}\}$

$Re\{\frac{1}{2} V_0 e^{j\phi} e^{j\omega t}\}$

$$\boxed{\frac{1}{2} V_0 e^{j\phi}} = \text{"Phasor"}$$

"complex exponential" $= e^{j\theta}$

$V(t) = \sim \boxed{\frac{V_0}{2} e^{j\phi_V}} e^{j\omega t} + \ldots$

$I(t) = \sim \boxed{\frac{I_0}{2} e^{j\phi_I}} e^{j\omega t} + \ldots$

$\omega, t \rightarrow$ same for everything

- "Phase vector"
- Encodes phase + magnitude
- Assumption: V_0, ω, ϕ are time-invariant aka constant

$\overline{e^{j\phi}} = e^{-j\phi}$

The phasor representation is a constant that contains the magnitude and phase information of the signal. The time-varying part of the signal does not need to be explicitly represented, because it is given by $e^{j\omega t}$, which is always implicit when using phasors. Phasors let us handle sinusoidal signals much more easily. They are powerful because they let us use DC-like (16A style) circuit analysis techniques, which we already know, to analyze circuits with sinusoidal voltages and currents.

Note: Can only use phasors with time-varying, sinusoidal inputs: ($v(t)$ ✓, V ✗)

Why are phasors important?

Remove time-dependency of capacitors & inductors

$i(t) = C \frac{dV(t)}{dt}$ $\quad$ $V(t) = L \frac{dI(t)}{dt}$

Impedance

$$\boxed{Z = \frac{\tilde{V}}{\tilde{I}}} \leftarrow \text{Phasor Representations}$$

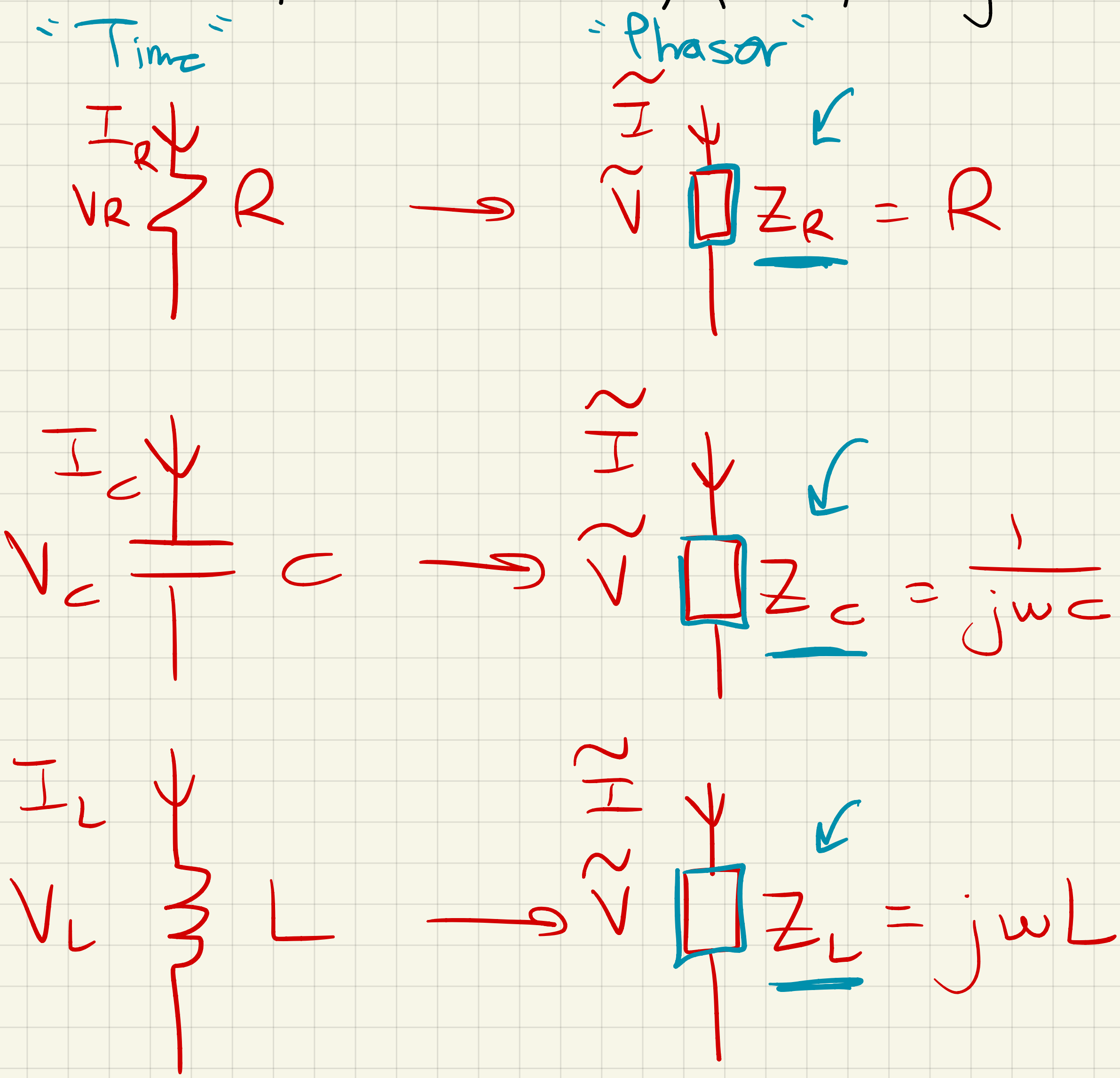
- $Z = R + \underline{jX}$, R = resistance, X = reactance

(Out of Scope)

- Concept of resistance generalized to capacitors and inductors

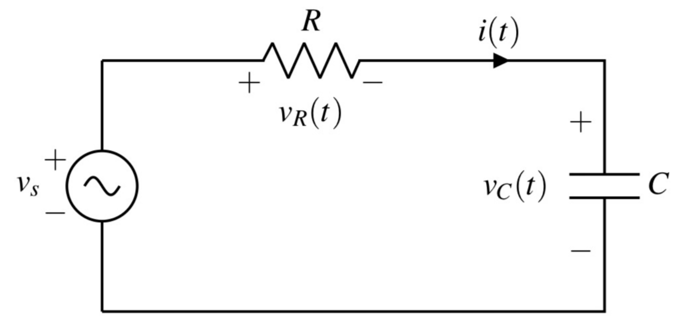
$V = IR \longrightarrow \tilde{V} = \tilde{I}Z$

Can treat every circuit element as a resistor, i.e. combine R, L, C in series, parallel, voltage dividers:



Q1: Phasor Analysis

The phasor analysis method consists of five steps. Consider the RC circuit below.



The voltage source is given by

$$v_s(t) = 12\sin\left(\omega t - \frac{\pi}{4}\right),$$

$= 12\cos\left(\omega t - \frac{\pi}{4} - \frac{\pi}{2}\right)$

$= 12\cos\left(\omega t - \frac{3\pi}{4}\right)$

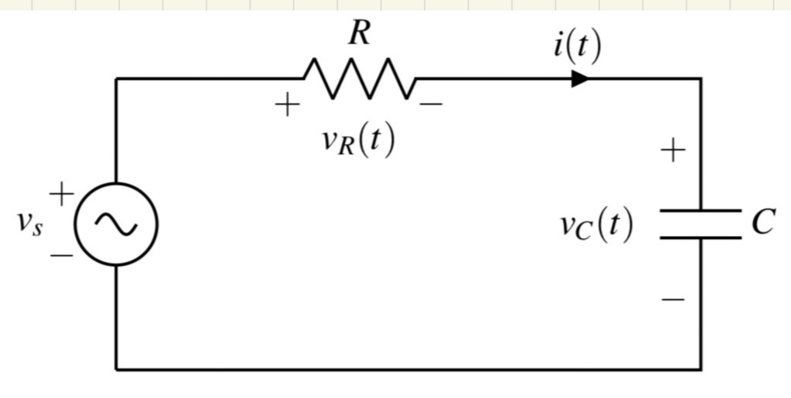
$\omega = 1000$ $\quad$ $\sqrt{3} \times 1000$ $\quad$ $1 \times 10^{-}$

with $\omega = 1 \times 10^3\ \frac{\text{rad}}{\text{s}}$, $R = \sqrt{3}\text{k}\Omega$, and $C = 1\,\mu\text{F}$.

Our goal is to obtain a solution for $i(t)$ with the sinusoidal voltage source $v_s(t)$.

(a) **Step 1: Write sources as exponentials:** $\widetilde{X}e^{j\omega t} + \overline{\widetilde{X}}e^{-j\omega t}$

All voltages and currents with known sinusoidal functions should be expressed in the standard exponential format. Convert $v_s(t)$ into a exponential and write down its phasor representation $\widetilde{V}_s$.



$12\cos\left(\omega t - \frac{3\pi}{4}\right)$

$\frac{V_0}{2} e^{j\phi}$

$\frac{12}{2} e^{j\left(-\frac{3\pi}{4}\right)}$

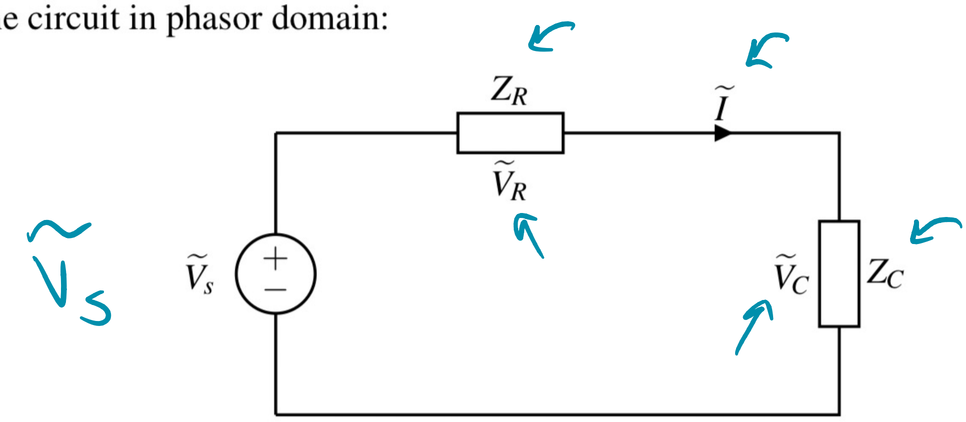
$$\underline{\widetilde{V}_s = 6 e^{-j\frac{3\pi}{4}}}$$

(b) **Step 2: Transform circuits to phasor domain**

The voltage source is represented by its phasor $\widetilde{V}_s$. The current $i(t)$ is related to its phasor counterpart $\widetilde{I}$ by

$$i(t) = \widetilde{I}e^{j\omega t} + \overline{\widetilde{I}}e^{-j\omega t}.$$

We redraw the circuit in phasor domain:



What are the impedances of the resistor, Z_R, and capacitor, Z_C? We sometimes also refer to this as the "phasor representation" of R and C.

$$Z_R = R$$

$$Z_C = \frac{1}{j\omega C}$$

(c) **Step 3: Cast the branch and element equations in phasor domain**

Use Kirchhoff's laws to write down a loop equation that relates all phasors in Step 2.

KVL: $\widetilde{V}_s = \widetilde{V}_R + \widetilde{V}_C$ $\qquad \widetilde{V} = \widetilde{I}Z$

$= \widetilde{I}Z_R + \widetilde{I}Z_C$

$\widetilde{V}_s = \widetilde{I}(Z_R + Z_C)$

(d) **Step 4: Solve for unknown variables**

Solve the equation you derived in Step 3 for $\widetilde{I}$ and $\widetilde{V}_C$. What is the polar form of $\widetilde{I}$ and $\widetilde{V}_C$? Polar form is given by $Ae^{j\theta}$, where A is a positive real number.

$$\widetilde{V}_s = 6e^{-j\frac{3\pi}{4}}$$

$$\underline{\widetilde{V}_s = \widetilde{I}\left(R + \frac{1}{j\omega C}\right)}$$

$R = \sqrt{3} \times 1000$

$C = 1 \times 10^{-6}$

$\omega = 1000$

$$\widetilde{I} = \left(\frac{1}{R + \frac{1}{j\omega C}}\widetilde{V}_s\right)\frac{j\omega C}{j\omega C}$$

$$\widetilde{I} = \frac{j\omega C}{1 + j\omega RC}\widetilde{V}_s$$

$$\widetilde{V}_c = \widetilde{I} Z_c = \text{''}\ \frac{1}{j\omega C} = \frac{1}{1 + j\omega RC}\widetilde{V}_s$$

$$\widetilde{I} = \frac{j(1000)(1 \times 10^{-6})}{1 + j\cancel{(1000)}(\sqrt{3} \times \cancel{1000})\cancel{(1 \times 10^{-6})}} 6e^{-j\frac{3\pi}{4}}$$

$$= \frac{j(1 \times 10^{-3})}{1 + \sqrt{3}j} 6e^{-j\frac{3\pi}{4}}$$

$$\boxed{j = e^{j\frac{\pi}{2}}}$$

$$= \frac{1 \times 10^{-3}}{1 + \sqrt{3}j} 6e^{-j\frac{\pi}{4}} \longrightarrow$$

$\angle \text{Top} - \angle \text{Bottom}$

Want: $\widetilde{I}$ in polar form: $\frac{|\widetilde{I}|}{2} e^{j\phi_I}$

Want: Magnitude of $\tilde{I}$

Phase of $\tilde{I}$

Magnitude: $|\tilde{I}| = \dfrac{|\text{Top}|}{|\text{Bottom}|}$

$$\tilde{I} = \frac{|6\times10^{-3}\, e^{-j\frac{\pi}{4}}|}{|\,1+\sqrt{3}j\,|} = \frac{6\times10^{-3}}{\sqrt{1^2+\sqrt{3}^2}} = \frac{6\times10^{-3}}{2}$$

$$I_m = 3\times10^{-3}$$

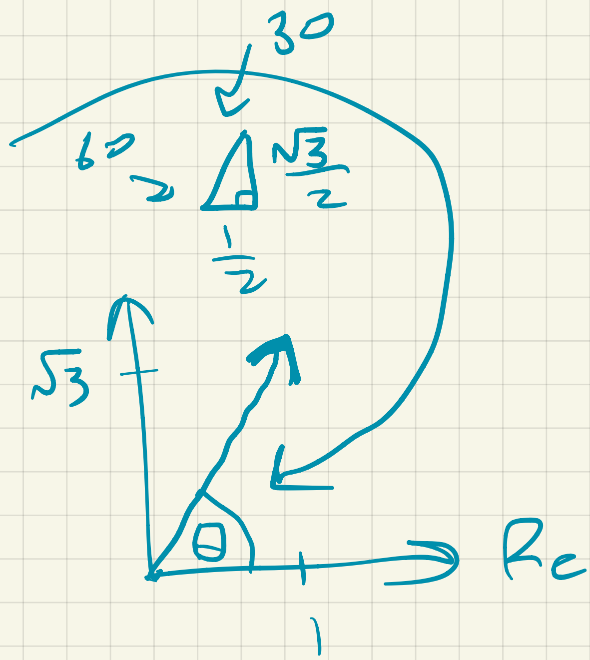
"Angle"

Phase: $\angle \text{Top} - \angle \text{Bottom}$

$$-\frac{\pi}{4} - 60^\circ$$

$$-\frac{\pi}{4} - \frac{\pi}{3}$$

$$= -\frac{7\pi}{12}$$



$$\tilde{I} = (3\times10^{-3})\, e^{-j\frac{7\pi}{12}} \quad \text{A}$$

$$\tilde{V}_c = 3\, e^{-j\frac{7\pi}{12}} \quad \text{V}$$

(e) **Step 5: Transform solutions back to time domain**

To return to time domain, we apply the fundamental relation between a sinusoidal function and its phasor counterpart. What is $i(t)$ and $v_C(t)$? What is the phase difference between $i(t)$ and $v_C(t)$?

$$V_0 \cos(\omega t + \phi) \longleftrightarrow \frac{1}{2} V_0 e^{j\phi}$$

$$I_0 \cos(\omega t + \phi) \longleftrightarrow \frac{1}{2} I_0 e^{j\phi}$$

$$I(t) = 2(3 \times 10^{-3}) \cos(\omega t + (-\tfrac{7\pi}{12}))$$

$$= 6 \times 10^{-3} \cos(\omega t - \tfrac{7\pi}{12}) \text{ A}$$

$$\boxed{I(t) = 6 \cos(\omega t - \tfrac{7\pi}{12}) \text{ mA}}$$

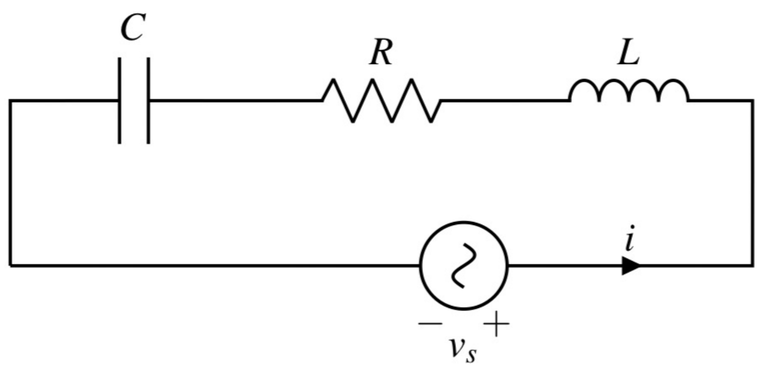
$$V_C(t) = 2(3) \cos(\omega t - \tfrac{7\pi}{12}) \text{ V}$$

$$\boxed{V_C(t) = 6 \cos(\omega t - \tfrac{7\pi}{12}) \text{ V}}$$

2. RLC Circuit Phasor Analysis

We study a simple RLC circuit with an AC voltage source given by

$$v_s(t) = B\cos(\omega t - \phi)$$

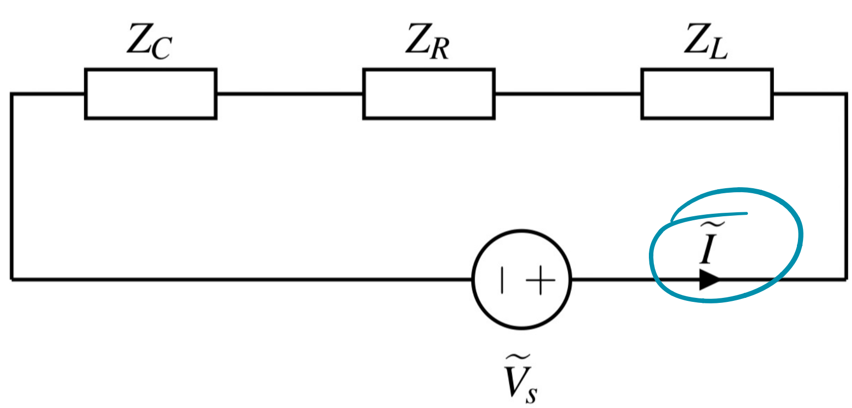


(a) Write out the phasor representation of $v_s(t)$, and the impedances of R, C, and L.

$$\tilde{V}_s = \frac{B}{2} e^{-j\phi}$$

(b) Now, we're going to redraw the circuit in the phasor domain.

Solve for $\tilde{I}$



Use Kirchhoff's laws to write down a loop equation relating the phasors.

$$Z_C = \frac{1}{j\omega L} \qquad Z_R = R$$

$$Z_L = j\omega L$$

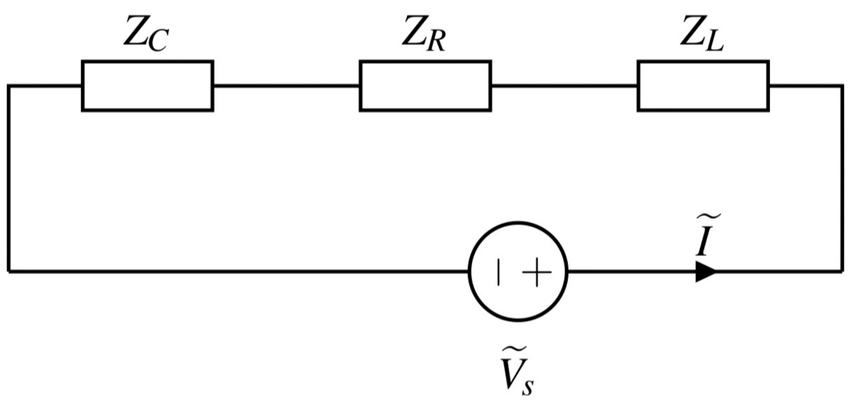
(c) Solve the equation in the previous step for the current $\tilde{I}$. What is the magnitude and phase of the polar form of $\tilde{I}$?

Hint: You'll need the following identities, which you can find in Note j:

- $|z_1/z_2| = |z_1|/|z_2|$
- $\angle(\frac{z_1}{z_2}) = \angle z_1 - \angle z_2$
- $\angle(a + jb) = \text{atan2}(b, a)$[3]

(b) Now, we're going to redraw the circuit in the phasor domain.

Solve for $\tilde{I}$



Use Kirchhoff's laws to write down a loop equation relating the phasors.

$$\tilde{V}_s = \tilde{I} Z_C + \tilde{I} Z_R + \tilde{I} Z_L$$

$$\tilde{V}_s = \tilde{I}(Z_C + Z_R + Z_L)$$

$$\tilde{I} = \frac{1}{\left[\frac{1}{j\omega C} + R + j\omega L\right.} \tilde{V}_s$$

$$R + j\omega L + \frac{1}{j\omega C}$$

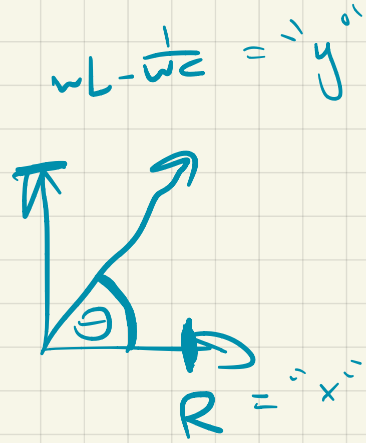
$$R + j\left(\omega L - \frac{1}{\omega C}\right)$$

$$\tilde{V}_s = \frac{B}{2} e^{j\phi}$$

$$\frac{1}{j\omega C} = -\frac{j}{\omega C}$$

$$|\tilde{I}| = \frac{B}{2}\left(\frac{1}{\sqrt{R^2 + \left(-\frac{1}{\omega C} + \omega L\right)^2}}\right)$$

$$\tilde{I} = \left(\frac{1}{R + j\left(\omega L - \frac{1}{\omega C}\right)}\right)\tilde{V}_s$$



$$= |\tilde{V}_s| \frac{|\text{Top}|}{|\text{Bottom}|}$$

$$\angle\tilde{I} = -\phi - \arctan\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)$$

$$\tan^{-1}\left(\frac{y}{x}\right)$$

$$j\omega L + \underbrace{\frac{1}{j\omega C}} = j\left(\omega L - \frac{1}{\omega C}\right)$$

$$\frac{1}{j\omega C}\left(\frac{j}{j}\right) = \frac{j}{j^2\omega C}$$

$$= -\frac{j}{\omega C}$$

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